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COMPUTATIONAL COMPLEXITY OF NORM-MAXIMIZATION

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This paper discusses the problem of maximizing a quasiconvex function ϕ over a convex polytope P in n-space that is presented as the intersection of a finite number of halfspaces. The problem is known to be **NP**-hard (for variable n) when ϕ is the p^{th} power of the classical p-norm. The present reexamination of the problem establishes **NP**-hardness for a wider class of functions, and for the p-norm it proves the **NP**-hardness of maximization over n-dimensional *parallelotopes* that are centered at the origin or have a vertex there. This in turn implies the **NP**-hardness of $\{-1,1\}$-maximization and $\{0,1\}$-maximization of a positive definite quadratic form. On the "good" side, there is an efficient algorithm for maximizing the Euclidean norm over an arbitrary *rectangular* parallelotope.

Introduction

We are concerned with the computational complexity of the decision problem that arises in attempting to maximize a quasiconvex function ϕ over an $\mathcal{H}$-presented polytope P in $\mathbb{R}^n$ (for variable n), where "$\mathcal{H}$-presented" means that the polytope is given as the intersection of a finite number m of closed halfspaces. For this problem, an instance consists of an $n \in \mathbb{N}$, an integral $\mathcal{H}$-presentation of $P \subset \mathbb{R}^n$, and an integer β, and the question is whether there exists $y \in P$ such that $\phi(y) \geq \beta$. The search for a maximum is simplified by the fact that P's vertex-set V contains a maximizing point, but complicated by the fact that the maximum possible cardinality of V is not bounded by any polynomial in n and m. In the case of greatest interest, there exists $p \in \mathbb{N}$ such that $\phi(x) = (\|x\|_p)^p = \Sigma_{i=1}^n |\xi_i|^p$ for each $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. In this case, **NP**-completeness has been established by Freund and Orlin [8] for $p = 2$ and by Mangasarian and Shiau [28] for general $p \in \mathbb{N}$.

After some geometric preliminaries in Section 1, our reexamination of norm-maximization leads to the following main conclusions, listed according to the sections in which they are established or discussed:

(Section 2) For the decision-theoretic version of the problem of maximizing a function over an $\mathcal{H}$-presented polytope, **NP**-hardness persists for a class of quasi-convex functions ϕ significantly broader than the p^{th} powers of p-norms — roughly speaking, for the ϕ whose level sets admit strictly inscribed full-dimensional paral-

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lelotopes. This includes, in particular, each function ϕ of the form $\Sigma_{i=1}^{k}\alpha_i\|x\|_{p_i}^{q_i}$, where the α_i's and q_i's are positive integers, each p_i is ∞ or a divisor of q_i, and not all p_i's are ∞. (When all p_i's are ∞, the maximization of ϕ can be accomplished in polynomial time by linear programming.) **NP**-hardness is established by a transformation from the known **NP**-complete problem, PARTITION [20].

(Section 2) It is conjectured that norm-maximization over $\mathcal{H}$-presented polytopes is **NP**-hard in all cases except those in which the unit ball is itself a polytope with a number of facets that is bounded by a polynomial in the dimension of the space. However, this probably cannot be proved by the present methods, for it is also conjectured that in a certain precise sense, almost no high-dimensional convex body admits an inscribed full-dimensional parallelotope.

(Section 3) For the p^{th} powers of p-norms, the known **NP**-hardness can be sharpened by restricting P to be either of the following very special sorts of polytope in $\mathbb{R}^n$:

a parallelotope that is given as a vector sum of the form $\Sigma_{i=1}^{m}[-x_i, x_i]$, where the m points x_i are linearly independent and their coordinates are all -1, 0 or 1;

an n-dimensional parallelotope that is centered at the origin or has a vertex there.

Here the **NP**-hardness is established by a series of transformations that begins with the known **NP**-complete problem, NOT-ALL-EQUAL 3-SAT [38].

(Section 4) The problems of $\{-1,1\}$-maximization and $\{0,1\}$-maximization of a positive definite quadratic form are **NP**-complete.

(Section 4) For maximizing the *Euclidean* norm over a *rectangular* parallelotope in $\mathbb{R}^n$ there is an algorithm that uses only n inner-product computations and $n-1$ comparisons.

The exposition assumes some familiarity with the classes **P** and **NP**, and with the rudiments of the theory of **NP**-completeness ([5], [20], [10]). A problem Y is said to be **NP**-*hard* if for each member X of **NP** there is a *transformation* of X to Y — that is, a deterministic polynomial-time algorithm which, when applied to an arbitrary instance of X, produces an equivalent instance of Y. (Here *equivalent* means merely that both are "yes" instances or both are "no" instances.) The problem Y is **NP**-*complete* if it is **NP**-hard and belongs to the class **NP**. The class of all **NP**-complete ⟨**NP**-hard⟩ problems is denoted here by **NPC** ⟨**NPH**⟩.

1. Polytopes, zonotopes, and parallelotopes

This section supplies the definitions and notation that are used in the rest of the paper.

A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is *quasiconvex* if for each pair of points $x, z \in \mathbb{R}^n$, it is true that

$$\sup_{y\in[x,z]} \phi(y) = \max\{\phi(x), \phi(z)\},$$

where $[x,z]$ denotes the line segment joining x and z. Equivalently, for each real λ the *level set* $\{x : \phi(x) \leq \lambda\}$ is convex.

For $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, the ∞-norm $\| \|_\infty$ is defined by

$$\|x\|_\infty = \max\{|\xi_1|, \ldots, |\xi_n|\}$$

and the *p-norm* is defined for $1 \leq p < \infty$ by

$$\|x\|_p = \left(\sum_{i=1}^{n} |\xi_i|^p \right)^{1/p}.$$

These norms, or certain positive linear combinations of powers of them, will be the quasiconvex (in fact, convex) functions of greatest concern here.

A *polytope* is a bounded subset of $\mathbb{R}^n$ (for some finite n) that is the intersection of a finite number of closed halfspaces. The *faces* of a polytope are its intersections with supporting hyperplanes. Prefixes indicate dimension, and the 0-faces, 1-faces and $(n-1)$-faces of an n-polytope are respectively its *vertices*, *edges* and *facets*.

Each polytope P is the convex hull of its finite vertex-set V, and it is well-known [17] that the maximum over P of any quasiconvex function is attained at some point of V. Thus when P is presented in terms of V, maximizing the norm over P presents no difficulties that are not already encountered in attempting to compute the norm at a particular point. However, the picture changes when the n-polytope P is presented as an intersection of m halfspaces, because then, although P's dimension is at most n and P has at most m facets, the number of vertices may be as large as

$$\binom{m - \lfloor (n+1)/2 \rfloor}{m-n} + \binom{m - \lfloor (n+2)/2 \rfloor}{m-n}$$

(see [29]). This sum is $O(m^{\lfloor n/2 \rfloor})$ for each fixed n, but for variable n and m it is not polynomially bounded. The exponential growth in the maximum size of V is reflected in the **NP**-hardness results proved here. We shall be concerned especially with n-parallelotopes, which have $2n$ facets and 2^n vertices.

Some definitions are required in order to deal with presentations of polytopes. An *$\mathcal{H}$-presentation* of a polytope P consists of integers m and n with $m > n \geq 1$, an $m \times n$ matrix A, and an m-vector b such that $P = \{x \in \mathbb{R}^n : Ax \leq b\}$. ($\mathcal{H}$ is for halfspace.) It is convenient to refer to P as an *$\mathcal{H}$-polytope*, to indicate that P is presented in this way. In dealing with such a presentation, we adopt the notational convention that $b = (\beta_1, \ldots, \beta_m)^T$ and $A = [\alpha_{ij}]$. When all the entries of A and b are integers ⟨rationals⟩, we speak of an *integer ⟨rational⟩ presentation.* The *size* L of an integer presentation is defined as the number of bits required for the natural binary encoding of the data — that is,

$$\begin{aligned} L = {} & (2 + \lceil \log m \rceil) + (2 + \lceil \log n \rceil) \\ & + (2mn + \sum_{\alpha_{ij} \neq 0} \lceil \log |\alpha_{ij}| \rceil) + (2m + \sum_{\beta_i \neq 0} \lceil \log |\beta_i| \rceil), \end{aligned}$$

where the logarithms are to the base 2. The size of a rational presentation is defined similarly, taking into account the denominators as well as the numerators of the input data. Note that each rationally presented polytope also admits an integer presentation. Note also that if the polytope is *full-dimensional* (of dimension n in $\mathbb{R}^n$) and the presentation is *irredundant* (the intersection

$$\bigcap_{i=1}^{m} \left\{ x = (\xi_1, \ldots, \xi_n)^T : \sum_{j=1}^{n} \alpha_{ij} \xi_j \leq \beta_i \right\}$$

is enlarged when any of the m halfspaces is omitted), then the hyperplanes

$$\left\{x = (\xi_1, \dots, \xi_n)^T : \sum_{j=1}^{n} \alpha_{ij}\xi_j = \beta_i\right\}$$

are precisely the affine hulls of the facets of P.

A *zonotope* is the vector sum (Minkowski sum) of a finite number of line segments. When the segments are $S_1, \dots, S_m$ and their centers are $c_1, \dots, c_m$, we have

$$\sum_{i=1}^{m} S_i = \left(\sum_{i=1}^{m} c_i\right) + \sum_{i=1}^{m}(S_i - c_i)$$

where the point $\Sigma_{i=1}^{m} c_i$ is the center of symmetry of the set $\Sigma_{i=1}^{m} S_i$ and each segment $S_i - c_i$ is centered at the origin. Hence it is convenient to define an $\mathcal{S}$-presentation ($\mathcal{S}$ for segment) of a zonotope in $\mathbf{R}^n$ as a sequence $(c; z_1, \dots, z_m)$ of points of $\mathbf{R}^n$, where c is the center and the z_i's are the ends of segments centered at the origin 0, with one end listed for each such segment. This sequence represents the zonotope

$$c + \sum_{i=1}^{m}[-z_i, z_i] = c + \left\{\sum_{i=1}^{m} \varepsilon_i z_i : |\varepsilon_i| \le 1 \text{ for all } i\right\}.$$

We speak of *integer* and *rational* $\mathcal{S}$-presentations (and define their sizes) in the natural way. Of course each zonotope is a polytope, but in general neither the vertices nor the facets of a zonotope are immediately accessible from an $\mathcal{S}$-presentation.

A zonotope $Z = c + \sum_{i=1}^{m}[-z_i, z_i]$ is called a *parallelotope* when the points $z_1, \dots, z_m$ are linearly independent. In contrast to the case of general zonotopes, the facial structure of a parallelotope is immediately accessible from an $\mathcal{S}$-presentation. In particular, the vertices of the m-parallelotope $c + \sum_{i=1}^{m}[-z_i, z_i]$ are the 2^m points of the form

$$c + \sum_{i=1}^{m} \varepsilon_i z_i \quad \text{with} \quad |\varepsilon_i| = 1 \text{ for all } i$$

and the facets are the $2m$ $(m-1)$-parallelotopes of the form

$$c - z_j + \sum_{i \ne j}[-z_i, z_i] \qquad \text{or} \qquad c + z_j + \sum_{i \ne j}[-z_i, z_i].$$

Also, the passage between $\mathcal{S}$-presentations and irredundant $\mathcal{H}$-presentations can be accomplished in polynomial time in the case of parallelotopes.

As the term is used here, a *body* is a compact convex set that has nonempty interior. A polytope P is *inscribed* (resp. *strictly inscribed*) in a body C if P's intersection with C's boundary contains (resp. consists of) P's vertex-set. When C is *rotund* (meaning that the boundary ∂C contains no line segment), each polytope P that is inscribed in C is strictly inscribed. When C is itself a polytope, the strict inscription of P in C implies that each facet of C includes at most one vertex of P and hence C has at least as many facets as P has vertices. In particular, if C is an n-parallelotope and $n \ge 3$, then no n-parallelotope P is strictly inscribed in C.

2. Convex functions and inscribed parallelotopes

Suppose that for each $n \in \mathbb{N}$, ϕ_n is a real-valued function whose domain is $\mathbb{R}^n$ and $\mathcal{P}_n$ is a collection of polytopes in $\mathbb{R}^n$. Let $\Phi = (\phi_1, \phi_2, \ldots)$ and $\mathfrak{P}=(\mathcal{P}_1,\mathcal{P}_2,\ldots)$. Then MAX$\Phi\mathfrak{P}$ will denote the following decision problem:

Instance: Positive integer n, integer β; integral $\mathcal{H}$-presentation of a member P of $\mathcal{P}_n$.

Question: Does there exist $x \in P$ such that $\phi_n(x) \geq \beta$?

Here the size of the instance is defined as the size of the presentation plus the size $2 + \lceil \log \beta \rceil$ of β.

The present paper is concerned with the following general problem:

For which sequences Φ and $\mathfrak{P}$ does the problem MAX$\Phi\mathfrak{P}$ belong to the class **P**? to **NP**? to **NPC**? to **NPH**?

The problem is too general to admit a meaningful general answer, but several subproblems are of interest. Note that, for a specific choice of Φ, an assertion MAX$\Phi\mathfrak{P} \in$ **NPH** is strengthened when the $\mathcal{P}_n$'s are replaced by smaller collections of polytopes, while an assertion MAX$\Phi\mathfrak{P} \in$ **P** or MAX$\Phi\mathfrak{P} \in$ **NP** is strengthened when the $\mathcal{P}_n$'s are replaced by larger collections of polytopes.

Theorem 1. *Suppose that each of the functions $\Phi_n : \mathbb{R}^n \to \mathbb{R}$ in the sequence Φ is quasiconvex, and that there is a polynomial-time algorithm which, for arbitrary input (n, x) with $n \in \mathbb{N}$ and $x \in \mathbb{Q}^n$, outputs $\phi_n(x)$. Then MAX$\Phi\mathfrak{P} \in$ **NP** for each choice of $\mathfrak{P}$.*

Proof. This is just a more general formulation of a well-known observation — see, e.g., [8] and [28]. For each $\mathcal{H}$-presentation of a polytope P in $\mathbb{R}^n$, the P-maximum of ϕ_n is attained at a vertex of P. When the presentation is rational, the vertex belongs to $\mathbb{Q}^n$ and its size is bounded by a polynomial in the size of the presentation. Each vertex v of P is the unique point of intersection of some n of the hyperplanes bounding the halfspaces that appear in the presentation, and the equations of these hyperplanes can be solved to find v. The "guessing algorithm" needed to establish membership in **NP** consists of guessing a set $\{H_1, \ldots, H_n\}$ of n hyperplanes of the mentioned sort. The "checking algorithm" processes each guess as follows:

(i) Determine whether the intersection $\cap_{i=1}^n H_i$ is empty or consists of more than one point; if either of these conditions is satisfied, stop; otherwise, find the unique point $w \in \cap_{i=1}^n H_i$.

(ii) Check to see whether w lies in the other halfspaces of the presentation (in which case it is a vertex of P), and whether $\phi_n(w) \geq \beta$; if both of these conditions are satisfied, return a "yes" answer. ∎

The conclusion of Theorem 1 continues to hold under an approximative assumption that is weaker than the requirement that $\phi_n(x)$ can be computed in polynomial time. It is enough to assume that $\phi_n(x)$ can be approximated in polynomial time and that the distance of $\phi_n(x)$ from any integer is either 0 or is bounded away from 0 by a positive rational of polynomial size. Here is a precise statement of the result:

Suppose that the functions ϕ_n in the sequence Φ are quasiconvex and satisfy the following separation condition: There is a polynomial π in L such that for each $n \in \mathbb{N}$, each $x \in \mathbb{Q}^n$ of size at most L, and each integer τ it is true that

$$\phi_n(x) = \tau \qquad \text{or} \qquad |\phi_n(x) - \tau| > 2^{-\pi(L)}.$$

Suppose also that there is a polynomial κ in L such that for each $x \in \mathbb{Q}^n$ of size at most L, $\phi_n(x)$ can be approximated in time $\kappa(L)$ with error less than $2^{-2\pi(L)}$. Then $MAX\Phi\mathfrak{P} \in \mathbf{NP}$ for each choice of $\mathfrak{P}$.

The proof is essentially unchanged, except that at the end, the approximation oracle is used to compute a rational number δ such that $|\phi_n(w) - \delta| \leq 2^{-2\pi(L)}$. Then we have

$$\phi_n(w) \geq \beta \quad \Leftrightarrow \quad \delta \geq \beta - 2^{-2\pi(L)}.$$

Theorem 2. *With hypotheses as in Theorem 1, suppose in addition that the sequence Φ is such that the number of vertices of any member P of $\cup_{n=1}^{\infty}\mathcal{P}_n$ is bounded by a polynomial in the size of P's presentation. Then $MAX\Phi\mathfrak{P} \in \mathbf{P}$.*

Proof. In this case, the vertex-set can be found in polynomial time (see [7], [40], [39]). Given $P \in \mathcal{P}_n$ and the threshold β, evaluate ϕ_n at the various vertices of P and thus decide whether $\max \phi_n P \geq \beta$. ∎

Theorem 3. *If each of the functions ϕ_n in the sequence Φ is concave and quadratic, then $MAX\Phi\mathfrak{P} \in \mathbf{P}$ for each choice of $\mathfrak{P}$.*

Proof. This follows from the fact that problems of convex quadratic minimization (subject to linear inequality constraints) can be solved in polynomial time by the ellipsoid method [24], and also by interior-point methods ([18], [30], [41]; see also [22]). ∎

The following theorem and its proof are due to Mangasarian and Shiau [28], a closely related result to Freund and Orlin [8], and weaker related results to Sahni [37] and Murty and Kabadi [31].

Theorem 4. [28] *For each positive integer p, the following problem is* **NP***-complete:*
Instance: Positive integers $\alpha_1, \ldots, \alpha_n$.
Question: Does there exist a point $x = (\xi_1, \ldots, \xi_n) \in \mathbb{Q}^n$ such that

$$-1 \leq \xi_i \leq 1 \quad \text{for} \quad 1 \leq i \leq n,$$

$$\sum_{i=1}^{n} \alpha_i \xi_i = 0,$$

and

$$\sum_{i=1}^{n} |\xi_i|^p \geq n.$$

Proof. The set defined by the inequality constraints is precisely $\{-1, 1\}^n$, the vertex-set of the cube $[-1, 1]^n$. Hence the following assertions are equivalent for each instance $(\alpha_1, \ldots, \alpha_n)$ of the problem:

(i) $(\alpha_1, \ldots, \alpha_n)$ is a "yes" instance;

(ii) there exists $(\xi_1, \ldots, \xi_n) \in \{-1, 1\}^n$ such that $\Sigma_{i=1}^n \alpha_i \xi_i = 0$;

(iii) the set $\{1, \ldots, n\}$ can be partitioned into two sets I and J such that $\Sigma_{i \in I} \alpha_i = \Sigma_{j \in J} \alpha_j$.

In other words, the problem of Theorem 4 is equivalent to Karp's **NP**-complete problem, PARTITION [20], which has the same instances as Theorem 4 and has as its question whether assertion (iii) is true. ∎

Corollary 5. *Suppose that p is a positive integer, and for each $n \in \mathbb{N}$ let ϕ_n be the p^{th} power of the p-norm on $\mathbb{R}^n$. Then*

$$\mathrm{MAX}(\phi_1, \phi_2, \ldots)\mathfrak{P} \in \mathbf{NPC}$$

when each $\mathcal{P}_n$ is the class of all polytopes centered at the origin in $\mathbb{R}^n$.

In the proof of Theorem 4, some of the intersections $[-1,1]^n \cap H$ are $(n-1)$-parallelotopes, but for $n \geq 3$ the affine type of the intersection varies with the choice of the positive sequence $\alpha_1, \ldots, \alpha_n$. For example, when $n = 3$ hexagons as well as parallelograms appear, and for larger values of n the intersections are not all zonotopes. See Naumann [32] for additional information about sections of n-cubes.

Although Theorems 4–5 and their proofs are extremely simple, three aspects seem worthy of further study:

(a) What is the geometric essence of the proof of Theorem 4?

(b) To what extent does the method of proof of Theorem 4 apply to functions more general than the p^{th} power of the p-norm?

(c) To what extent can the class $\mathfrak{P}$ of polytopes in Corollary 5 be further restricted?

Question (b) is discussed later in the present section, question (c) in Section 3. The following theorem answers question (a) by showing that the geometric essence of the proof of Theorem 4 is the fact that a suitable level set of the objective function admits a strictly inscribed parallelotope.

Theorem 6. *Suppose that for each n, $\mathcal{P}_n$ is the class of all centrally symmetric $n-1$-polytopes in $\mathbb{R}^n$, and suppose that the following are given:*

(i) a quasiconvex function $\phi_n : \mathbb{R}^n \to \mathbb{R}$;

(ii) a number λ_n whose size is bounded by a polynomial in n;

(iii) a rational $\mathcal{S}$-presentation $(c; z_1, \ldots, z_n)$ of an n-parallelotope that is strictly inscribed in the body $\{x \in \mathbb{R}^n : \phi_n(x) \leq \lambda_n\}$, the size of the presentation being bounded by a polynomial in n.

Then with $\mathfrak{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots)$, the problem $\mathrm{MAX}(\phi_1, \phi_2, \ldots)\mathfrak{P}$ is $\mathbf{NP}$-hard. If, in addition,

(iv) there is an algorithm which, for an arbitrary point $q \in \mathbb{Q}^n$, decides in polynomial time whether $\phi_n(q) \geq \lambda_n$,

then the problem $\mathrm{MAX}(\phi_1, \phi_2, \ldots)\mathfrak{P}$ is $\mathbf{NP}$-complete.

Proof. Let B denote the $n \times n$ matrix whose j^{th} column lists the coordinates of the point z_j $(1 \leq j \leq n)$. Then B is invertible and the size of B^{-1} is bounded by a polynomial in n. For each point $x = (\xi_1, \ldots, \xi_n)^T \in \mathbb{R}^n$ there is a unique $y = (\eta_1, \ldots, \eta_n)^T \in \mathbb{R}^n$ such that

$$x = c + \eta_1 z_1 + \cdots + \eta_n z_n.$$

This is equivalent to saying that $x = c + By$, or $y = B^{-1}x - B^{-1}c$. Now consider an arbitrary instance $(\alpha_1, \ldots, \alpha_n)$ of PARTITION, and let P consist of all points $x = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ for which

$$-1 \leq \eta_i \leq 1 \quad \text{for} \quad 1 \leq i \leq n,$$

$$\sum_{i=1}^{n} \alpha_i \eta_i = 0.$$

Equivalently, if e denotes the n-vector all of whose coordinates are 1 and R is the row-vector $(\alpha_1, \ldots, \alpha_n)B^{-1}$, then P is defined by the system of inequalities

$$-e + B^{-1}c \leq B^{-1}x \leq e + B^{-1}c$$

in conjunction with the equality $Rx = Rc$. Since the $\mathcal{S}$-presented parallelotope $(c; z_1, \ldots, z_n)$ is strictly inscribed in the level set $\{x : \phi_n(x) \leq \lambda_n\}$, the following two statements are equivalent for an arbitrary point x satisfying the above system of inequalities:

(a) $\phi_n(x) \geq \lambda_n$;

(b) the set $\{1, \ldots, n\}$ can be partitioned into two sets I and J such that $\eta_i = -1$ for all $i \in I$ and $\eta_j = 1$ for all $j \in J$.

Under the additional condition that $\Sigma_{i=1}^{n} \alpha_i \eta_i = 0$, (b) is equivalent to requiring that the set $\{1, \ldots, n\}$ can be partitioned into two sets I and J such that $\Sigma_{i \in I} \alpha_i = \Sigma_{j \in J} \alpha_j$. Thus we have a transformation of PARTITION to $\mathrm{MAX}(\phi_1, \phi_2, \ldots)\mathfrak{P}$.

(Since the size of λ_n is bounded by a polynomial in n, the above instance of $\mathrm{MAX}(\phi_1, \phi_2, \ldots)\mathfrak{P}$ can be constructed in polynomial time. Recall that an instance of $\mathrm{MAX}(\phi_1, \phi_2, \ldots)\mathfrak{P}$ consists of the dimension n, a bound β, and an integral $\mathcal{H}$-presentation of P; in particular, the functions ϕ_i are not part of the input but are given in advance.) ∎

Corollary 7. *Suppose that for each n, p_n is a positive integer and $\mathcal{P}_n$ is the class of all polytopes in $\mathbb{R}^n$. Let*

$$\Phi = (\| \|_{p_1}{}^{p_1}, \| \|_{p_2}{}^{p_2}, \ldots) \quad \textit{and} \quad \mathfrak{P} = (\mathcal{P}_1, \mathcal{P}_2, \ldots).$$

Then the problem $\mathrm{MAX}\Phi\mathfrak{P}$ *is* **NP**-*hard. If, in addition, the p_n's are bounded by a polynomial in n, then* $\mathrm{MAX}\Phi\mathfrak{P}$ *is* **NP**-*complete.*

Proof. Observe that with $\phi_n = \| \|_{p_n}{}^{p_n}$ and $\lambda_n = n$, the cube $[-1, 1]^n$ is strictly inscribed in the body

$$\{x \in \mathbb{R}^n : \|x\|_{p_n} \leq n^{1/p_n}\} = \{x \in \mathbb{R}^n : \phi_n(x) \leq \lambda_n\}.$$

Hence the assumptions (i), (ii) and (iii) of Theorem 6 are satisfied. If, in addition, the sequence $(p_1, p_2, \ldots)$ is uniformly bounded by a polynomial in n, then for rational vectors $q = (\kappa_1, \ldots, \kappa_n)$ and positive integers β it can be decided in polynomial time whether

$$\|q\|_{p_n}{}^{p_n} = \sum_{i=1}^{n} |\kappa_i|^{p_n} \geq \beta.$$

Hence the stated conclusions follow from Theorem 6. ∎

We believe that norm-maximization over general $\mathcal{H}$-presented polytopes is likely to be **NP**-hard for any sequence of norms except those in which the unit ball is a polytope with a number of facets that is bounded by a polynomial in the dimension of the space. In this exceptional case, the problem of maximizing the norm over a rationally $\mathcal{H}$-presented polytope P can be solved in polynomial time by means of linear programming, using either the ellipsoid method [21] or an interior-point method [19]. In particular (as noted in [28]) a polynomial-time maximization algorithm for the ∞-norm consists of maximizing each coordinate functional and its negative over P, and noting that the maximum of the numbers obtained in this way is the maximum of the norm on P. Here the unit ball is a polytope with only $2n$ facets. On the other hand, the unit ball with respect to the 1-norm, though a polytope, has an exponential number of facets, and the **NP**-hardness result does apply to the 1-norm.

Concerning the existence of inscribed parallelotopes in situations more general than that of Corollary 7, we can offer only the following.

Theorem 8. *Suppose that the function $\phi : \mathbf{R}^n \to \mathbf{R}$ satisfies the following three conditions:*

(i) ϕ is quasiconvex and lower semicontinuous;

(ii) ϕ is not bounded above on any line through the origin;

(iii) for each point $x = (\xi_1, \ldots, \xi_n)$, the value of $\phi(x)$ is unchanged when any coordinate ξ_i of x is replaced by its negative.

Then ϕ attains its minimum value at the origin 0. For each $\lambda > \phi(0)$ and for each point y whose coordinates are all nonzero, some positive multiple of y is a vertex of an n-parallelotope that is centered at the origin and inscribed in the body $B_\lambda = \{x : \phi(x) \leq \lambda\}$.

Proof. Condition (iii) implies that $\phi(-x) = \phi(x)$ for all $x \in \mathbf{R}^n$. From this in conjunction with conditions (i) and (ii) it follows that for each line L through the origin 0,

$$\sup \phi L = \infty \qquad \text{and} \qquad \inf \phi L = \phi(0).$$

For each $\lambda > \phi(0)$, the set B_λ is certainly closed and convex, and if it were unbounded condition (iii) could be used to produce a line through 0 on which ϕ is bounded. Since that contradicts (ii), the set B_λ is a body. Now for $y = (\eta_1, \ldots, \eta_n)$ of the sort described, let the positive multiplier μ be such that $\mu y \in \partial B$. Then the set

$$P_y = \{x = (\xi_1, \ldots, \xi_n) : |\xi_i| \leq \mu\eta_i \text{ for all } i\}$$

is an n-parallelotope centered at the origin, and it follows from condition (iii) that P_y is inscribed in B_λ. ∎

Corollary 9. *Suppose that for $i = 1, \ldots, k$, α_i and q_i are positive integers and p_i is ∞ or a positive divisior of q_i. For each $n \in \mathbf{N}$, define*

$$\phi_n(x) = \sum_{i=1}^{k} \alpha_i \|x\|_{p_i}^{q_i} \qquad \textit{for all } x \in \mathbf{R}^n,$$

and let $\Phi = (\phi_1, \phi_2, \ldots)$. If all the p_i's are ∞, then $\mathrm{MAX}\Phi\mathfrak{P} \in \mathbf{P}$ when each $\mathcal{P}_n$ is the collection of all polytopes in $\mathbf{R}^n$. In all other cases, $\mathrm{MAX}\Phi\mathfrak{P} \in \mathbf{NPC}$ when $\mathcal{P}_n$ is the collection of all $(n-1)$-polytopes centered at the origin in $\mathbf{R}^n$.

Proof. Since objective functions are convex, membership in **NP** follows from Theorem 1. If there is an i for which $2 \leq p_i < \infty$, then the level sets of the objective functions are rotund and hence the inscribed parallelotopes guaranteed by Theorem 8 are strictly inscribed. If all p_i's are equal to 1, the level sets of the objective functions are just balls with respect to the 1-norm, and again there are strictly inscribed parallelotopes. In these cases it follows with the aid of Theorem 6 that $\mathrm{MAX}\Phi\mathfrak{P} \in \mathbf{NPH}$.

In the remaining case, all p_i's are ∞ and each objective function is a positive combination of positive powers of the ∞-norm. Hence maximizing the objective function over a polytope P is equivalent to maximizing the ∞-norm. That was discussed in the paragraph preceding Theorem 8. ∎

We end this section with some additional remarks about inscribed parallelotopes. It is known that each body in $\mathbb{R}^2$ admits many inscribed rectangles and inscribed rhombi, and at least one inscribed square [42] [4]. Each body C in $\mathbb{R}^3$ admits an inscribed 3-parallelotope whose volume is at least two-ninths that of C [3], but there is a 3-dimensional body that does not admit any inscribed *rectangular* 3-parallelotope [2]. It would be interesting to know what can be said about the existence of inscribed parallelotopes in higher-dimensional bodies. Note that if a centrally symmetric n-body C admits an inscribed n-parallelotope, then it admits one whose center coincides with C's. Note also that if each $(n-1)$-body admits an inscribed parallelotope, then each *centrally symmetric* n-body C admits many inscribed n-parallelotopes; indeed, each hyperplane that intersects C's interior but misses C's center c contains a facet of an n-parallelotope that is inscribed in C and centered at c. In particular, each symmetric 4-body admits an inscribed 4-parallelotope. Nevertheless, we conjecture that when the dimension n is large enough most symmetric n-bodies do not admit inscribed n-parallelotopes. To make the conjecture precise, let B_n denote the unit ball of Euclidean n-space, and for each $\tau > 1$ let $\mathcal{K}(n,\tau)$ denote the collection of all symmetric n-bodies K such that K is centered at the origin and $B_n \subseteq K \subseteq \tau B_n$. With respect to the Hausdorff distance, $\mathcal{K}(n,\tau)$ is a compact metric space. Let

$$\mathcal{K}'(n,\tau) = \{K \in \mathcal{K}(n,\tau) : K \text{ admits an inscribed } n\text{-parallelotope}\}.$$

Then $\mathcal{K}'(n,\tau)$ is an F_σ subset of $\mathcal{K}(n,\tau)$, and we make the following

Conjecture 10. *For each sufficiently large n and for each $\tau > 1$, the complement of $\mathcal{K}'(n,\tau)$ is a dense subset of $\mathcal{K}(n,\tau)$.*

Now recall that for each n there exists a τ_n such that each n-body centered at the origin is linearly equivalent to some member of $\mathcal{K}(n,\tau_n)$ [27] [26]. In view of this fact, and of the Baire category theorem, proof of the conjecture would establish our claim that "most" symmetric bodies do not admit inscribed parallelotopes.

Before returning to norm-maximization, we mention two other open problems concerning the existence of inscribed polytopes. Pucci [36] claimed to prove that each 3-body admits an inscribed regular octahedron. Hadwiger, Larman and Mani [14] showed that several of the claims in Pucci's argument are incorrect, but the existence of inscribed regular octahedra is still unsettled. Grünbaum [13] asked whether every n-body admits an inscribed polytope that is a translate of the vector sum of an n-simplex and its negative. He noted that several authors had established an affirmative answer for $n = 2$, where the polytopes in question are the affinely regular hexagons. However, the problem is open for all $n \geq 3$.

3. p-Norms

The **NP**-hardness proofs of Theorems 4 and 6 are short, but for each dimension n they use polytopes of a variety of affine types — the central $(n-1)$-sections of n-parallelotopes. In the present section we show that maximizing $\| \|_p^p$ over an $\mathcal{H}$-polytope in $\mathbb{R}^n$ is **NP**-hard even when the polytope is an n-parallelotope centered at the origin or having a vertex there. Since, in the case of parallelotopes, it is easy to move back and forth between $\mathcal{H}$-presentations and $\mathcal{V}$-presentations, we actually work with the following problems $[-1,1]\text{PARMAX}_p$ and $[0,1]\text{PARMAX}_p$, defined for each positive integer p:

Instance: Positive integers n and γ; n linearly independent integer vectors $x_1, \ldots, x_n$ in $\mathbb{R}^n$.

Question for $[-1,1]\text{PARMAX}_p$: Is the maximum of $\| \|_p^p$ on the parallelotope $\Sigma_{i=1}^n[-x_i, x_i]$ at least γ?

Question for $[0,1]\text{PARMAX}_p$: Is the maximum of $\| \|_p^p$ on the parallelotope $\Sigma_{i=1}^n[0, x_i]$ at least γ?

Theorem 11. *Each of the problems* $[-1,1]\text{PARMAX}_p$ *and* $[0,1]\text{PARMAX}_p$ *is polynomially transformable to the other.*

Proof. In each of the transformations below, it is assumed that the space $\mathbb{R}^n$ is canonically embedded in the space $\mathbb{R}^{n+1}$, and that $u_{n+1} = (0, \ldots, 0, 1)^T \in \mathbb{R}^{n+1}$.

Consider an arbitrary instance $(n, \gamma; x_1, \ldots, x_n)$ of $[0,1]\text{PARMAX}_p$, and let $P = \Sigma_{i=1}^n[0, x_i]$. Set

$$\delta = 2^p\gamma + 1,$$

$$y_i = x_i \text{ for } 1 \le i \le n, \qquad y_{n+1} = u_{n+1} + \sum_{i=1}^n x_i.$$

Then the given instance of $[0,1]\text{PARMAX}_p$ is equivalent to the instance $(n+1, \delta; y_1, \ldots, y_n, y_{n+1})$ of $[-1,1]\text{PARMAX}_p$. Indeed, for each nonzero vertex v of P it is true that $2v + u_{n+1}$ is a vertex of the parallelotope $\Sigma_{i=1}^{n+1}[-y_i, y_i]$, and that

$$\|2v + u_{n+1}\|_p^p = 2^p\|v\|_p^p + 1.$$

From this it follows that $(n, \gamma; x_1, \ldots, x_n)$ is a "yes" instance of $[0,1]\text{PARMAX}$ if and only if there exists $w \in u_{n+1} + 2P$ such that $\|w\|_p^p \ge \delta$. Further, it follows from the symmetry properties of the norm and of the parallelotope $\Sigma_{i=1}^{n+1}[-y_i, y_i]$ that the maximum of the norm on this parallelotope is attained at a vertex of the set $u_{n+1} + 2P$. That establishes the stated claim.

Now suppose, on the other hand, that $(n, \gamma; x_1, \ldots, x_n)$ is an instance of $[-1,1]\text{PARMAX}_p$, and let $P = \Sigma_{i=1}^n[-x_i, x_i]$. Set

$$\alpha = 2^{(p+2)L} \quad \text{and} \quad \delta = \gamma + \alpha,$$

where L is the size of encoding of the input data. Set

$$y_i = 2x_i \text{ for } 1 \le i \le n, \quad y_{n+1} = \alpha u_{n+1} - \sum_{i=1}^n x_i,$$

and $Q = \Sigma_{i=1}^{n+1}[0, y_i]$. Then the given instance of $[-1,1]\mathrm{PARMAX}_p$ is equivalent to the instance $(n+1, \delta; y_1, \ldots, y_n, y_{n+1})$ of $[0,1]\mathrm{PARMAX}_p$. To justify this claim, note that each vertex of Q belongs to one of the two parallel facets

$$P - y_{n+1} + \alpha u_{n+1} \quad \text{and} \quad P + \alpha u_{n+1}$$

and that $P + \alpha u_{n+1}$ is Q's intersection with the hyperplane $\{x = (\xi_1, \ldots, \xi_{n+1}) : \xi_{n+1} = \alpha\}$. Observe, further, that each point $x \in P$ is equal to $\Sigma_{i=1}^{n} \lambda_i x_i$ for some $\lambda_1, \ldots, \lambda_n \in [-1,1]$, and with $y_i = (\eta_1^{(i)}, \ldots, \eta_n^{(i)})^T$ we have

$$\begin{aligned} \|x\|_p^p = \left\| \sum_{i=1}^{n} \lambda_i x_i \right\|_p^p &= \sum_{j=1}^{n} \left| \sum_{j=1}^{n} \lambda_j \eta_j^{(i)} \right|^p \\ &\le \sum_{j=1}^{n} \sum_{i=1}^{n} \left| \eta_j^{(i)} \right|^p \le n^2 2^{pL} \le 2^{(p+1)L} < \frac{1}{2}\alpha. \end{aligned}$$

Since $P - y_{n+1} + \alpha u_{n+1} \subset 2P$, it follows that all vertices of Q that are farthest from the origin must belong to the facet $P + \alpha u_{n+1}$. Hence the given instance of $[-1,1]$PARMAX is a "yes" instance if and only if there is a vector $q \in Q$ such that $\|q\|_p^p \ge \gamma + 2^{(p+2)L}$. Thus we have transformed $[-1,1]\mathrm{PARMAX}_p$ to $[0,1]\mathrm{PARMAX}_p$. ∎

In the sequel, we use only the second of the two transformations described in the proof of Theorem 11. The first is included to illustrate the fact that, in constructing the points that generate the respective parallelotopes, control over the magnitudes of coordinates is much firmer in the first transformation than in the second. That may be relevant to a question about strong **NP**-completeness that is mentioned at the end of this section.

Our next goal is to establish the **NP**-hardness of the problem $[-1,1]\mathrm{PARMAX}_p$. We begin with a transformation from the following problem 3SPLIT:

Instance: A finite set M and a collection $\mathscr{C}$ of subsets of M, each member of $\mathscr{C}$ being of cardinality 2 or 3

Question: Can M be partitioned into two sets, both of which intersect each member of $\mathscr{C}$?

The **NP**-completeness of 3SPLIT was established (in a slightly different but clearly equivalent formulation) by Schaefer [38], who called the problem NOT–ALL–EQUAL 3-SAT. We use the term 3SPLIT for brevity. The **NP**-completeness of a closely related problem, HYPERGRAPH 2-COLORING, was recognized by Lovász [25].

Our first transformation of 3SPLIT establishes the **NP**-hardness of the problem of maximizing $\| \ \|_p^p$ over m-dimensional parallelotopes centered at the origin in $\mathbb{R}^n$. Since $m < n$ is permitted here, additional transformations are needed to arrive at the problem $[-1,1]\mathrm{PARMAX}_p$.

Theorem 12. *For each fixed positive integer p, the following problem* $\mathrm{PAR}\{-1,0,1\}\mathrm{MAX}_p$ *is* **NP***-complete:*

Instance: Positive integers $n, m \le n$ and α, and m linearly independent vectors $x_1, \ldots, x_m$ in $\{-1,0,1\}^n$.

Question: Do there exist $\varepsilon_1, \ldots, \varepsilon_m$ *in* $\{-1,1\}$ *such that* $\|\Sigma_{i=1}^m \varepsilon_i x_i\|_p^p \geq \alpha$*?*

Proof. Clearly the problem belongs to **NP**. Simply guess $\varepsilon_1, \ldots, \varepsilon_m$, and let $(\xi_1, \ldots, \xi_n)^T = \Sigma_{i=1}^m \varepsilon_i x_i$, so that

$$\left\|\sum_{i=1}^m \varepsilon_i x_i\right\|_p^p = \sum_{j=1}^n |\xi_j|^p.$$

Then check whether $\Sigma_{j=1}^n |\xi_j|^p \geq \alpha$.

Now consider an instance $(M, \mathscr{C})$ of 3SPLIT, and assume without loss of generality that $M = \{1, \ldots, m\}$. Let the 3-pointed members of $\mathscr{C}$ be $C_1, \ldots, C_r$, let the 2-pointed members be $C_{r+1}, \ldots, C_{r+s}$, and set $n = 3r + s + m$. We are going to describe a polynomial-time construction that yields m vectors $x_1, \ldots, x_m$ in $\{-1,0,1\}^n$ — one vector for each member of the underlying set M — such that for each partition of M into two sets G and H, the following is true:

$$\left(\left\|\sum_{g\in G} x_g - \sum_{h\in H} x_h\right\|_p\right)^p \geq (3^p + 2)r + 2^p s + m$$

if and only if

both G and H intersect each member of $\mathscr{C}$.

Thus for each instance of 3SPLIT we have an equivalent instance of PAR$\{-1,0,1\}$MAX$_p$, with

$$n = 3r + s + m \qquad \text{and} \qquad \gamma = (3^p + 2)r + 2^p s + m.$$

That will establish the **NP**-hardness of PAR$\{-1,0,1\}$MAX$_p$.

It is convenient to think of the n coordinates $(\xi_1^{(i)}, \ldots, \xi_n^{(i)})$ of each point x_i as divided into successive blocks of length $3r$, s and m. The last m coordinates of the points $x_1, \ldots, x_m$ are chosen to assure that these points will be linearly independent. Specifically,

$$\xi_{3r+s+i}^{(i)} = 1, \quad \text{and otherwise } \xi_k^{(i)} = 0 \text{ for } 3r + s < k \leq n.$$

The first $3r$ coordinates of x_i are determined by the 3-pointed members of $\mathscr{C}$ to which the point i of M belongs, and the next s coordinates of x_i are determined by the 2-pointed members of $\mathscr{C}$ to which i belongs. For $1 \leq k \leq r$, the coordinates $\xi_{3k-2}^{(i)}, \xi_{3k-1}^{(i)}$, and $\xi_{3k}^{(i)}$ are 0 when $i \notin C_k$, and when $i \in C_k$ these coordinates are 1 or -1 as specified below. For $1 \leq k \leq s$, the coordinate $\xi_{3r+k}^{(i)}$ is 0 when $i \notin C_{3r+k}$, and when $i \in C_{3r+k}$ this coordinate is 1 or -1 as specified below.

For $1 \leq j \leq 3r$, the coordinate $\xi_j^{(i)}$ of x_i is specified to be

$$\begin{gathered} 0 \text{ if } i \notin C_{\lceil j/3 \rceil}, \\ -1 \text{ if } i \in C_{\lceil j/3 \rceil} \text{ and condition } (*) \text{ holds}, \\ 1 \text{ if } i \in C_{\lceil j/3 \rceil} \text{ and condition } (*) \text{ fails}, \end{gathered}$$

where the condition $(*)$ is that

$$\begin{array}{lll} \text{either} & j \equiv 0 \pmod 3 \quad \text{and} \quad C_{\lceil j/3 \rceil} \subseteq [1,i] \\ \text{or} & j \equiv 1 \pmod 3 \quad \text{and} \quad C_{\lceil j/3 \rceil} \subseteq [i,m] \\ \text{or} & \left\{ \begin{array}{l} j \equiv 2 \pmod 3 \quad \text{and} \quad C_{\lceil j/3 \rceil} \text{ includes both} \\ \text{members of } M \text{ that are greater than } i \text{ and} \\ \text{members of } M \text{ that are less than } i \end{array} \right. \end{array}$$

(Here $[1,i] = \{h \in M : 1 \leq h \leq i\}$ and $[i,m] = \{h \in M : i \leq h \leq m\}$.)

For $3r < j \leq n$, the coordinate $\xi_j^{(i)}$ of x_i is specified to be

$$\begin{array}{c} 0 \text{ if } i \notin C_{j-2r}, \\ -1 \text{ if } i \in C_{j-2r} \subseteq [i,m], \\ 1 \text{ if } i \in C_{j-2r} \subseteq [1,i]. \end{array}$$

(Since $|C_{j-2r}| = 2$ in this case, when $i \in C_{j-2r}$ it must be true that either $C_{j-2r} \subseteq [i,m]$ or $C_{j-2r} \subseteq [1,i]$.)

Before proving that the vectors $x_1, \ldots, x_m$ have the desired property, we illustrate the construction of the x_i's by an example. With $p = 2$, suppose that

$$M = \{1,2,3,4,5\} \text{ and } \mathscr{C} = \{\{1,2,4,\},\{3,4,5\},\{1,5\},\{2,4\}\}.$$

Then $m = 5,\ r = 2,\ s = 2,$

$$n = 3r + s + m = 13 \text{ and } \alpha = (3^p + 2)r + 2^p s + m = 35.$$

The points $y_1, \ldots, y_5$ of $\{-1,0,1\}^{13}$ are the columns of the matrix below, where the division into blocks is indicated.

$$\begin{array}{ccccc} -1 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & -1 \\ \hline -1 & 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 & 0 \\ \hline 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}$$

Now consider an arbitrary sequence $\varepsilon_1, \ldots, \varepsilon_m$ in $\{-1,1\}$, and let

$$x = (\xi_1, \ldots, \xi_n)^T = \sum_{i=1}^{m} \varepsilon_i x_i.$$

If $1 \leq k \leq r$ and $C_k = \{a,b,c\}$ with $a < b < c$, then

$$\begin{pmatrix} \xi_{3k-2} \\ \xi_{3k-1} \\ \xi_{3k} \end{pmatrix} = \varepsilon_a \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} + \varepsilon_b \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + \varepsilon_c \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

From this follows that

$$3 \leq |\xi_{3k-2}|^p + |\xi_{3k-1}|^p + |\xi_{3k}|^p \leq 3^p + 2,$$

with equality on the right if and only if $\varepsilon_a \neq \varepsilon_b$ or $\varepsilon_a \neq \varepsilon_c$. Similarly, if $1 \leq k \leq s$ and $C_{r+k} = \{d, e\}$ with $d < e$, then

$$\xi_{3r+k} = \varepsilon_d(-1) + \varepsilon_e(1),$$

from which it follows that $0 \leq |\xi_{3r+k}|^p \leq 2^p$, with equality on the right if and only if $\varepsilon_d \neq \varepsilon_e$.

To complete the proof, verify that the following four statements are equivalent:

(i) $$\|x\|_p^p \geq (3^p + 2)r + 2^p s + m;$$

(ii) $$|\xi_{3k-2}|^p + |\xi_{3k-1}|^p + |\xi_{3k}|^p = 3^p + 2 \quad \text{for each } k \in [1, r]$$

and $$|\xi_{3r+k}|^p = 2^p \quad \text{for each } k \in [1, s];$$

(iii) for each 3-pointed member $\{a, b, c\}$ of $\mathscr{C}$ with $a < b < c$,

$$\varepsilon_a \neq \varepsilon_b \quad \text{or} \quad \varepsilon_a \neq \varepsilon_c,$$

and for each 2-pointed member $\{d, e\}$ of $\mathscr{C}$ with $d < e$, $\varepsilon_d \neq \varepsilon_e$.

(iv) each member of $\mathscr{C}$ intersects both the set $G = \{i : \varepsilon_i = -1\}$ and the set $H = \{i : \varepsilon_i = -1\}$. ∎

Note that in the case of PAR$\{-1,0,1\}$MAX$_p$, the $\{-1, 0, 1\}$ notation indicates that only -1, 0 and 1 are permitted as coordinates of the vectors x_i arising in an instance of the problem. However, in the cases of $[-1,1]$PARMAX$_p$ and $[0,1]$PARMAX$_p$, the $[-1,1]$ and $[0,1]$ merely indicate how the parallelotopes are related to the origin.

We want next to transform PAR$\{-1,0,1\}$MAX$_p$ to $[-1,1]$PARMAX$_p$. To do this in the Euclidean case ($p = 2$), it might be simplest to proceed as follows: (i) find the $(n-m)$-dimensional subspace S of $\mathbf{Q}^n$ that is orthogonal to the linear span of the m x_i's appearing in PAR$\{-1,0,1\}$MAX$_p$; (ii) find a basis $y_1, \ldots, y_m$ for S that is approximately orthonormal; (iii) for an appropriately chosen multiplier μ, let $y_i = \mu x_{i-m}$ for $m < i \leq n$, and then work with the n-parallelotope $\Sigma_{i=1}^n[-y_i, y_i]$. However, in order to deal with an arbitrary positive integer p we transform PAR$\{-1,0,1\}$MAX$_p$ to $[-1,1]$PARMAX$_p$ in two stages, where the intermediate stage involves a full-dimensional zonotope that is not in general a parallelotope.

Theorem 13. *For each positive integer p, the following problem $[-1,1]$ZONMAX$_p$ is* **NP**-*complete:*

Instance: Positive integers n, $t \geq n$, and β, and vectors $y_1, \ldots, y_t$ in $\mathbf{R}^n$ such that $y_1, \ldots, y_n$ is the standard basis for $\mathbf{R}^n$ and each of $y_{n+1}, \ldots, y_t$ belongs to $\{-(n+1)^p, 0, (n+1)^p\}^n$.

Question: Do there exist $\varepsilon_1, \ldots, \varepsilon_t$ in $\{-1, 1\}$ such that $\|\Sigma_{i=1}^t \varepsilon_i y_i\|_p^p \geq \beta$?

Proof. Membership in **NP** is obvious. To establish **NP**-hardness, consider an arbitrary instance $(n, m, \alpha; x_1, \ldots, x_m)$ of PAR$\{-1,0,1\}$MAX$_p$ and let $y_1, \ldots, y_n$ be the standard basis for $\mathbf{R}^n$. Set

$$t = n + m, \quad \mu = n + 1, \quad \beta = \mu^p \alpha, \quad y_i = \mu x_{i-n} \quad \text{for } n < i \leq t.$$

We claim that $(n, m, \alpha; x_i, \ldots, x_m)$ is a "yes" instance of PAR$\{-1,0,1\}$MAX$_p$ if and only if $(n, t, \beta; y_1, \ldots, y_t)$ is a "yes" instance of $[-1,1]$ZONMAX$_P$.

If $(n, m, \alpha; x_1, \ldots, x_m)$ is a "yes" instance of PAR$\{-1,0,1\}$MAX$_p$ there exist $\delta_1, \ldots, \delta_m$ in $\{-1,1\}$ such that

$$\left\|\sum_{i=1}^{m} \delta_i x_i\right\|_p^p \geq \alpha,$$

whence

$$\left\|\sum_{i=n+1}^{t} \delta_i y_i\right\|_p^p \geq \mu^p \alpha = \beta.$$

Since the function $\| \ \|_p^p$ is convex and the point $\Sigma_{i=n+1}^t \delta_i y_i$ is the average of the 2^n points of the form $\Sigma_{i=1}^n \varepsilon_i y_i + \Sigma_{i=n+1}^t \delta_i y_i$ with $\varepsilon_1, \ldots, \varepsilon_n \in \{-1,1\}$, there is a point of this form for which

$$\left\|\sum_{i=1}^{n} \varepsilon_i y_i + \sum_{i=n+1}^{t} \delta_i y_i\right\|_p^p \geq \left\|\sum_{i=n+1}^{t} \delta_i y_i\right\|_p^p.$$

Hence $(n, t, \beta; y_1, \ldots, y_t)$ is a "yes" instance of $[-1,1]$ZONMAX$_p$.

If, on the other hand, $(n, m, \alpha; x_1, \ldots, x_m)$ is a "no" instance of PAR$\{-1,0,1\}$MAX$_p$, then for each choice of $\varepsilon_1, \ldots, \varepsilon_t \in \{-1,1\}$ it is true that $\|\Sigma_{i=n+1}^t \varepsilon_i z_i\|^p \leq \alpha - 1$. But then

$$\left\|\sum_{i=1}^{t} \varepsilon_i y_i\right\|_p^p \leq n^p + \mu^p(\alpha - 1) = \beta - (\mu^p - n^p) < \beta$$

and hence $(n, t, \beta; y_1, \ldots, y_t)$ is a "no" instance of $[-1,1]$ZONMAX$_p$. That completes the proof. ∎

The following lemma will aid in proving Theorem 15.

Lemma 14. *Suppose that p, n, t and ψ are positive integers with $n < t$, and let $u_1, \ldots, u_t$ denote the standard basis for $\mathbb{R}^t$. With $\mathbb{R}^n$ canonically embedded in $\mathbb{R}^t$, suppose that $y_i = z_i = u_i$ for $1 \leq i \leq n$ and that $y_{n+1}, \ldots, y_t$ are points of $\mathbb{R}^n$, each of p-norm at most ψ. Suppose that δ and ε are positive real numbers with*

$$\varepsilon(t-n) \leq 2^{1/p} - 2^{1/2p} \quad \text{and} \quad \varepsilon(t-n) \leq \frac{\delta}{2p(t\psi)^{p-1}}$$

and let $z_i = y_i + \varepsilon u_i$ for $n < i \leq t$. Let $\mu(Y)$ and $\mu(Z_\varepsilon)$ denote the maximum of the p^{th} power of the p-norm on the zonotopes Y and Z_ε respectively, where

$$Y = \sum_{i=1}^{t} [-y_i, y_i] \subset \mathbb{R}^n \subset \mathbb{R}^t \quad \text{and} \quad Z_\varepsilon = \sum_{i=1}^{t} [-z_i, z_i] \subset \mathbb{R}^t.$$

Then

$$\mu(Y) \leq \mu(Z_\varepsilon)$$

and if $\mu(Y) \geq 2$ it is true also that

$$\mu(Z_\varepsilon) \leq \mu(Y) + \delta.$$

Proof. The first inequality follows from the fact that the p-norm of any point of $\mathbb{R}^t$ increases whenever one of its coordinates is increased in absolute value while the other coordinates are unchanged. For the second inequality, assume that $\mu(Y) \geq 2$ and let v be a vertex of Z_ε such that $\|v\|^p = \mu(Z_\varepsilon)$. Then there exist $\lambda_1, \ldots, \lambda_t$ in $\{-1, 1\}$ such that

$$v = \sum_{i=1}^{t} \lambda_i z_i = \sum_{i=1}^{t} \lambda_i y_i + \varepsilon \sum_{i=n+1}^{t} \lambda_i u_i.$$

Set

$$\sigma = \left\| \sum_{i=1}^{t} \lambda_i y_i \right\| \leq \sum_{i=1}^{t} \|y_i\| \leq t\psi$$

and note that $\|\Sigma_{i=n+1}^{t} \lambda_i u_i\| = t - n$. Then

$$2 \leq \mu(Z_\varepsilon) = \|v\|^p \leq (\sigma + \varepsilon(t-n))^p.$$

From this in conjunction with the first condition on the choice of ε, it follows that

$$\sigma \geq 2^{1/p} - \varepsilon(t-n) \geq 2^{1/2p}.$$

A routine computation shows that if

$$\sigma \geq 2^{1/2p} \quad \text{and} \quad 0 \leq \tau \leq 2^{1/p} - 2^{1/2p}$$

then

$$(\sigma + \tau)^p \leq \sigma^p + 2p\sigma^{p-1}\tau.$$

Hence this last inequality holds when $\tau = \varepsilon(t-n)$, and we have

$$(\sigma + \varepsilon(t-n))^p \leq \sigma^p + 2p\sigma^{p-1}\varepsilon(t-n).$$

To complete the proof that $\mu(Z_\varepsilon) \leq \mu(Y) + \delta$, note that $\sigma^p \leq \mu(Y)$ and apply the second condition on the choice of ε to the second summand on the right. ∎

Theorem 15. *For each positive integer p, the problems* $[-1,1]\text{PARMAX}_p$ *and* $[0,1]\text{PARMAX}_p$ *are* **NP**-*complete.*

Proof. In view of Theorems 11–13, it suffices to transform $[-1,1]\text{ZONMAX}_p$ to $[-1,1]\text{PARMAX}_p$. For notational simplicity, we work with the variant of $[-1,1]\text{PARMAX}_p$ in which the generating points $z_1, \ldots, z_n$ are permitted to have rational coordinates while the threshold β is an integer. This is permissible, for if $(n, \beta; z_1, \ldots, z_n)$ is an instance of this problem and λ is the product of the denominators of the coordinates of the z_i's, then the instances $(n, \beta; z_1, \ldots, z_n)$ and $(n, \lambda^p\beta; \lambda x_1, \ldots, \lambda x_n)$ are equivalent.

Now consider an arbitrary instance $(n, t, \beta; y_1, \ldots, y_t)$ of $[-1,1]$ZONMAX$_p$. If $t = n$, it is already an instance of $[-1,1]$PARMAX$_p$. When $t > n$, proceed as follows:

(i) Set

$$\psi = (n+1)^{p+1} > \max\{\|y_1\|, \ldots, \|y_t\|\},$$

where the inequality follows from the fact that each of the n coordinates of each point y_i is at most $(n+1)^p$ in absolute value.

(ii) Set $\delta = \frac{1}{2}$, and choose a positive rational

$$\varepsilon < \min\{2^{1/p} - 2^{1/2p}, \delta/(2pn^{p-1}\psi^{p-1})\}.$$

This can be done in polynomial time.

(iii) Form the points $z_1, \ldots, z_t$ as described in Lemma 14. The sequence $(n, \beta; z_1, \ldots, z_t)$ describes an instance of $[-1,1]$PARMAX$_p$.

(iv) By Lemma 14,

$$\mu(Y) \leq \mu(Z_\varepsilon) \leq \mu(Y) + \frac{1}{2}$$

We do not know that $\mu(Z_\varepsilon)$ is an integer. However, both $\mu(Y)$ and β are integers, and hence $\mu(Y) \geq \beta$ if and only if $\mu(Z_\varepsilon) \geq \beta$. Thus we have transformed the instance of $[-1,1]$ZONMAX$_p$ to an equivalent instance of the variant of $[-1,1]$PARMAX$_p$. ∎

Though PARTITION and 3SPLIT are both **NP**-complete, the former can be solved by a *pseudopolynomial* algorithm — one which, for each fixed upper bound B on the integers in the sequence that is to be partitioned, does admit a polynomial upper bound on running time (with the bound depending on B). In other words, the **NP**-hardness of PARTITION depends on the admissibility of instances involving integers that are very large relative to the underlying combinatorics. That is not true of 3SPLIT or $\{-1,0,1\}$PARMAX$_p$ or $[-1,1]$PARMAX$_p$. In the terminology of Garey and Johnson [9,10], these three problems are **NP**-complete in the strong sense while PARTITION is not. We do not know whether $[0,1]$PARMAX$_p$ is **NP**-complete in the strong sense.

Problem 16. *For a positive integer p, is the problem* $[0,1]$*PARMAX*$_p$ **NP**-*complete in the strong sense?*

In this connection, the reader is invited once more to compare the two transformations used in the proof of Theorem 11.

4. Euclidean norms

As was mentioned in connection with Theorem 3, there are polynomial-time algorithms for maximizing a concave quadratic function over an $\mathcal{H}$-polytope. For a survey concerning the much more difficult problem of maximizing a convex function, see [33]. As we saw in Section 3, the latter problem is **NP**-hard, even for the very special case in which the objective function is the square of the Euclidean norm and the polytope is a full-dimensional parallelotope that is centered at the origin or has a vertex there. However, the next result shows that the standard Euclidean norm can be maximized in polynomial time over an arbitrary *rectangular* parallelotope.

Theorem 17. *For maximizing the Euclidean norm over a rectangular parallelotope in* $\mathbf{R}^n$*, there is an algorithm that uses at most* n *inner-product computations and* $n-1$ *comparisons.*

Proof. If $(c; x_1, \ldots, x_m)$ is an $\mathscr{S}$-presentation of a rectangular parallelotope P in $\mathbf{R}^n$, then $m \leq n$ and the x_i's are pairwise orthogonal. Set $y = c - \Sigma_{i=1}^m x_i$, and let $s_i = 2x_i$ for $i = 1, \ldots, m$. Set $S_0 = \{0\}$ and for $1 \leq k \leq m$ define

$$S_k = \left\{ \sum_{i=1}^k \lambda_i s_i : \ \lambda_1, \ldots, \lambda_k \in [0,1] \right\}.$$

Then y is a vertex of P, the s_i's are pairwise orthogonal, and $P = y + S_m$.

Here is the algorithm:

begin
  $q \leftarrow y$;
  for $i \leftarrow 1$ **until** n **do**
    if $\langle q + s_i, q + s_i \rangle > \langle q, q \rangle$ **then** $q \leftarrow q + s_i$;
  `print "The norm is maximized by the vertex"` q
end.

Let $y_0 = y$, and for $1 \leq k \leq n$ let y_k denote the value of q after the k^{th} pass through the **do** loop. Then y_k is of course a vertex of the k-parallelotope $y + S_k$, and we claim that y_k maximizes the norm over $y + S_k$. That is obviously true when $k = 0$. For the inductive step from $k-1$ to k, note that the vertices of $y + S_k$ are precisely the points of the form $y + v$ or $p + v + s_k$ where v belongs to the vertex-set V_{k-1} of S_{k-1}. For each $v \in V_{k-1}$, we have

$$\begin{aligned}\langle y + v + s_k, y + v + s_k \rangle &= \langle y + v, y + v \rangle + \langle s_k, s_k \rangle + 2\langle y + v, s_k \rangle \\ &= \langle y + v, y + v \rangle + \langle s_k, s_k \rangle + 2\langle y, s_k \rangle,\end{aligned}$$

where the last step uses the fact that $\langle v, s_k \rangle = 0$. From this it follows that for each $v, w \in V_{k-1}$,

$$\langle y + v, y + v \rangle \geq \langle y + w, y + w \rangle$$
$$\text{if and only if}$$
$$\langle y + v + s_k, y + v + s_k \rangle \geq \langle y + w + s_k, y + w + s_k \rangle.$$

Since y_{k-1} maximizes the norm over $y + S_{k-1}$, we conclude that y_{k-1} or $y_{k-1} + s_k$ maximizes the norm over $y + S_k$, and that leads to the desired conclusion. ∎

We turn finally, to the subject of *pseudoboolean programming*, which is concerned with the optimization of real- or rational-valued functions over the vertex-set $\{0,1\}^n$ of the n-cube $[0,1]^n$. Because of the importance of this subject, and the wide range of problems that can be naturally formulated in these terms, it is of interest to identify classes of functions that can be maximized over $\{0,1\}^n$ in polynomial time. Some such classes of quadratic functions are identified in [35], [1], [16], and [6]. Other discussions of quadratic pseudoboolean programming appear in [23], [34], and [33].

In addition to identifying classes of quadratic functions for which pseudoboolean maximization is easy, it is of interest to identify classes for which it is **NP**-hard. Hammer and Simeone [15] show that such a class is formed by functions of the form

$x^T Ax$, where A is an upper triangular matrix that has at most one negative entry in each row. Here we do the same for $x^T Ax$ when the matrix A is positive definite. In this case, the function $x^T Ax$ is just the square of a norm with respect to which $\mathbf{R}^n$ is linearly isometric to the standard Euclidean n-space.

Theorem 18. *The following problems* POSDEF$\{-1,1\}$MAX *and* POSDEF$\{0,1\}$MAX *are* **NP**-*complete:*

Instance: Positive integers n and λ; positive definite symmetric $n \times n$ matrix A with integer entries.

Question for POSDEF$\{-1,1\}$MAX*: Does there exist a $\{-1,1\}$-vector x such that $x^T Ax \geq \gamma$?*

Question for POSDEF$\{0,1\}$MAX*: Does there exist a $\{0,1\}$-vector x such that $x^T Ax \geq \gamma$?*

Proof. Membership in **NP** is obvious. To establish **NP**-hardness, we describe a transformation of the $[-1,1]$PARMAX$_2$ (resp. $[0,1]$PARMAX$_2$) problem of Theorem 15 to POSDEF$\{-1,1\}$MAX (resp. POSDEF$\{0,1\}$MAX). An instance of a PARMAX$_2$ problem consists of positive integers n and β, and n linearly independent integer vectors $x_1, \ldots, x_n$ in $\mathbf{R}^n$. Let A denote the Gramian matrix formed by the inner products of the x_i's — i.e., $A = [\alpha_{ij}]$, where $\alpha_{ij} = x_i^T x_j$. Clearly A is a symmetric matrix whose entries are integers, and it is well known that A is positive definite when the x_i's are linearly independent. Hence $(n, \beta; A)$ is an instance of a POSDEFMAX problem. Each point x of $\mathbf{Q}^n$ has a unique representation in the form $x = \Sigma_{i=1}^n \lambda_i x_i$ with rational λ_i's, and

$$\|x\|^2 = x^T x = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i^T x_j)\lambda_i \lambda_j.$$

This is equal to $y^T Ay$ with $y = (\lambda_1, \ldots, \lambda_n)^T$. Hence $\|x\|^2$ attains a value $\geq \beta$ for λ_i's in $\{-1,1\}$ (resp. $\{0,1\}$) if and only if $y^T Ay$ attains a value $\geq \beta$ on the vertex-set of the cube $\{-1,1\}^n$ (resp. $[0,1]^n$). Thus the PARMAX$_2$ problems have been transformed to the respective POSDEFMAX problems and the latter, like the former, are **NP**-complete. ∎

An alternative proof of Theorem 18 can be drawn from the papers [11] and [12]. [12] shows by a transformation from PARTITION that the following problem, SIMPLEX–WIDTH, is **NP**-hard:

Instance: an n-simplex T in $\mathbf{R}^n$, a positive integer ρ.

Question: In the Euclidean norm for $\mathbf{R}^n$, is the square of the width of T less than or equal to ρ?

(The *width* of a body C is the minimum of the distances between pairs of parallel supporting hyperplanes of C.) [11] outlines an argument showing that the problems POSDEF$\{0,1\}$MAX and SIMPLEX–WIDTH are *polynomially equivalent* in the sense that there is a one-to-one transformation (polynomial-time in both directions) between the instances of one problem and those of the other. Hence the **NP**-hardness of POSDEF$\{0,1\}$MAX follows from that of PARTITION by way of SIMPLEX–WIDTH. This provides a different proof of Theorem 18 and also, in view of the following paragraphs, a different proof of Theorem 15 for the case $p = 2$.

By a construction similar to that in the first part of the proof of Theorem 11, POSDEF$\{0,1\}$MAX can be transformed to POSDEF$\{-1,1\}$MAX. In closing, we shall describe a transformation of POSDEF$\{-1,1\}$MAX to $[-1,1]$PARMAX$_2$. Consider an instance $(n, \lambda; A)$ of POSDEF$\{-1,1\}$MAX, where A is an $n \times n$ symmetric positive definite matrix with integer entries. In polynomial time we can compute a permutation matrix J, a lower triangular matrix L, a diagonal matrix D, and an upper triangular matrix U such that $JA = LDU$. Let us assume (without loss of generality) that J is the $n \times n$ identity matrix. Since A is symmetric and the LDU factorization is unique, we have $L = U^T$. Thus the computation of the factorization is essentially Gaussian elimination. Observe that the entries of D and U are rationals whose numerators and denominators are polynomially bounded in the size of the input.

Since A is positive definite, the diagonal entries of D are all positive. Let S denote the diagonal matrix obtained from D by taking the square roots of D's diagonal entries. The entries of S may of course be irrational, but it will become clear that suitable rational approximations are available. For simplicity we first work with real arithmetic. Let $W = SU$. Then $A = W^T W$, and with $C = [-1,1]^n$ the following four statements are equivalent:

there exists $x \in C$ such that $x^T Ax \geq \lambda$;

there exists $x \in C$ such that $(Wx)^T(Wx) \geq \lambda$;

there exists $y \in WC$ such that $y^T y \geq \lambda$;

there exists $y \in P$ such that $\|y\|_2^2 \geq \lambda$, where P is the parallelotope given by $P = \Sigma_{i=1}^n [-w_i, w_i]$ and the w_i's are the columns of the matrix W.

The preceding discussion shows that in real arithmetic, POSDEF$\{-1,1\}$MAX transforms polynomially to $[-1,1]$PARMAX. To deal with the binary (Turing machine) model of computation, observe that $x^T Ax$ is an integer for each vertex of the cube $C = [-1,1]^n$. Hence the following two statements are equivalent whenever $0 \leq \varepsilon < 1$:

there exists $x \in C$ such that $x^T Ax \geq \lambda$;

there exists $x \in C$ such that $x^T Ax \geq \lambda - \varepsilon$.

From this equivalence it follows that rounding the entries of S to a suitable number of significant digits (bounded by a polynomial in the size of the input) can be accomplished without destroying the transformation of POSDEF$\{-1,1\}$MAX to $[-1,1]$PARMAX. Hence the two problems are polynomially equivalent for the binary model of computation.

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